Different styles of writing solutions to systems of linear equations:

or "$\forall$" = "for any"

- parametric form: $\{(x_1, x_2, x_3) = (3t-2,\ s+t,\ -5+4s+t) : s,t \in \mathbb{R}\}$

"parameters" = what we are free to assign free variables to be

- alternatively: $\begin{cases} x_1 = 3t-2 \\ x_2 = s+t \\ x_3 = -5+4s+t \end{cases}, \ \forall\ s,t \in \mathbb{R}$

- parametric vector form $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3t-2 \\ s+t \\ -5+4s+t \end{pmatrix} : s,t \in \mathbb{R} \right\}$

better way, in my mind

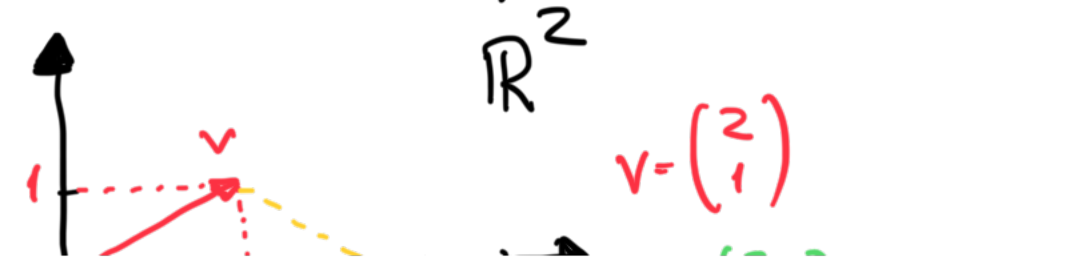
I like "$\forall$" more

a solution is a vector

All the above applies if the system is consistent, i.e. has at least one solution. If it has no solutions, we call the system inconsistent, and write the solution set

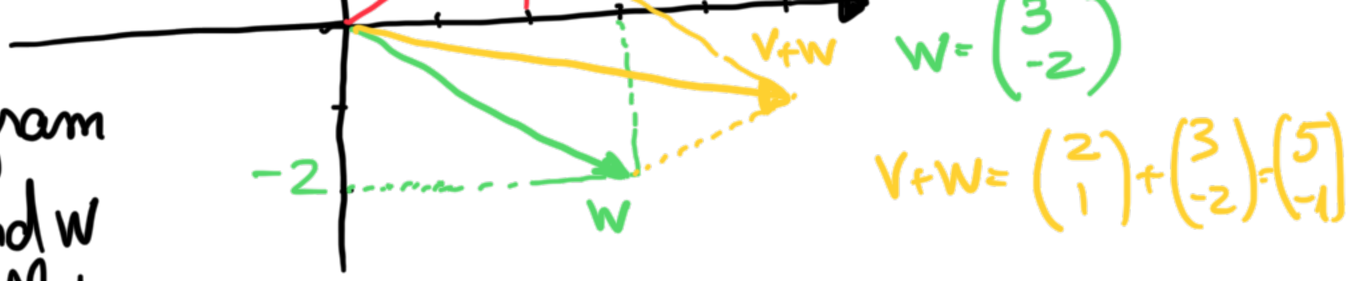
$\emptyset$ = empty set

Last time : $v \in \mathbb{R}^n$ = n-dimensional space



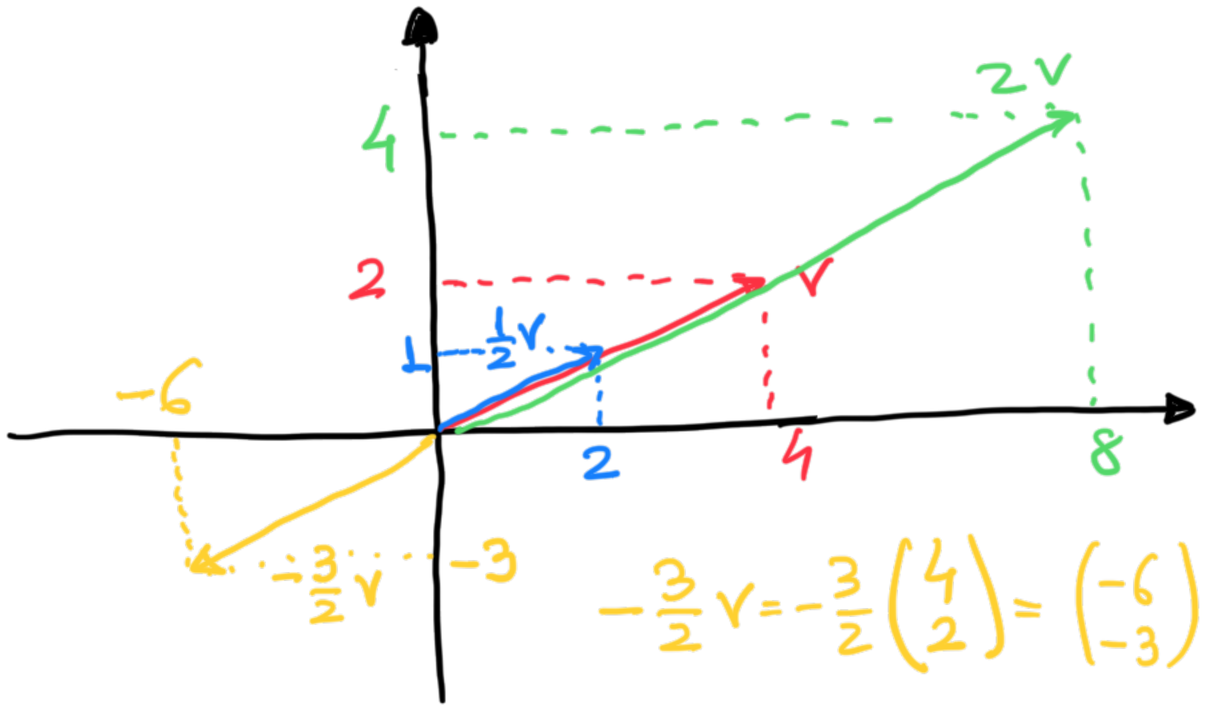
- addition :

draw the parallelogram bounded by v and w and then v+w will be the diagonal vector



- scalar multiplication:

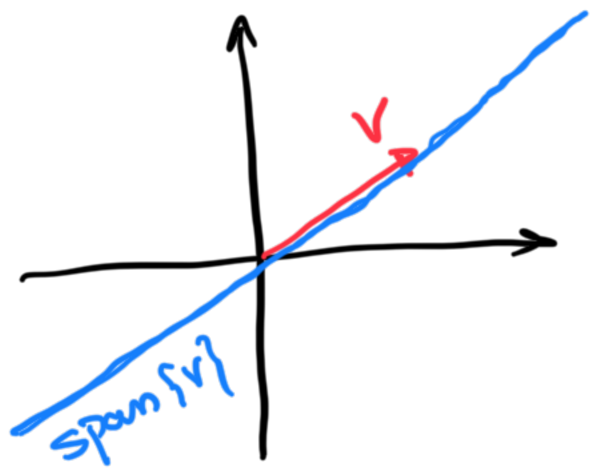
  λv has the same direction as v, but λ determines its size



Linear combination (of v & w) is any $\lambda v + \mu w$, $\lambda, \mu \in \mathbb{R}$

→ given vectors $v_1, \dots, v_m$, a linear combination is any vector of the form $c_1 v_1 + \dots + c_m v_m$, $c_1, \dots, c_m \in \mathbb{R}$

→ $\text{Span}\{v_1, \dots, v_m\}$ = set of all linear combinations



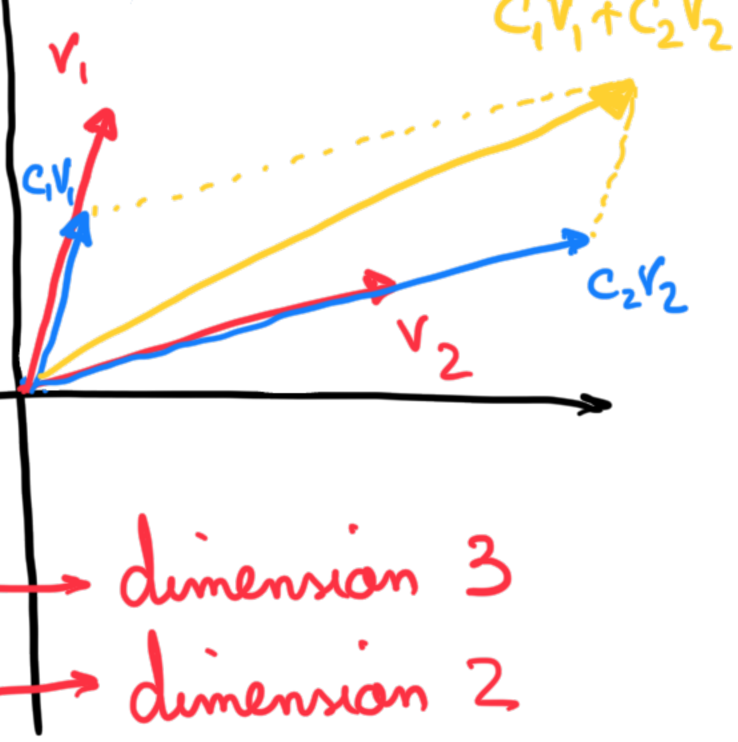
- $m=1$, $\text{span}\{v\} = \{cv, c \in \mathbb{R}\}$

  upshot: span{v} is a line, except when $v=0$ in which case the span is a point (the origin)

- $m=2$, $\text{span}\{v_1, v_2\} = \{c_1 v_1 + c_2 v_2, c_1, c_2 \in \mathbb{R}\}$

  upshot: span$\{v_1, v_2\}$ is a plane,

except when v_1 and v_2 are collinear in which case it is a line, except when v_1, v_2 are 0 in which case the span is a point (the origin)

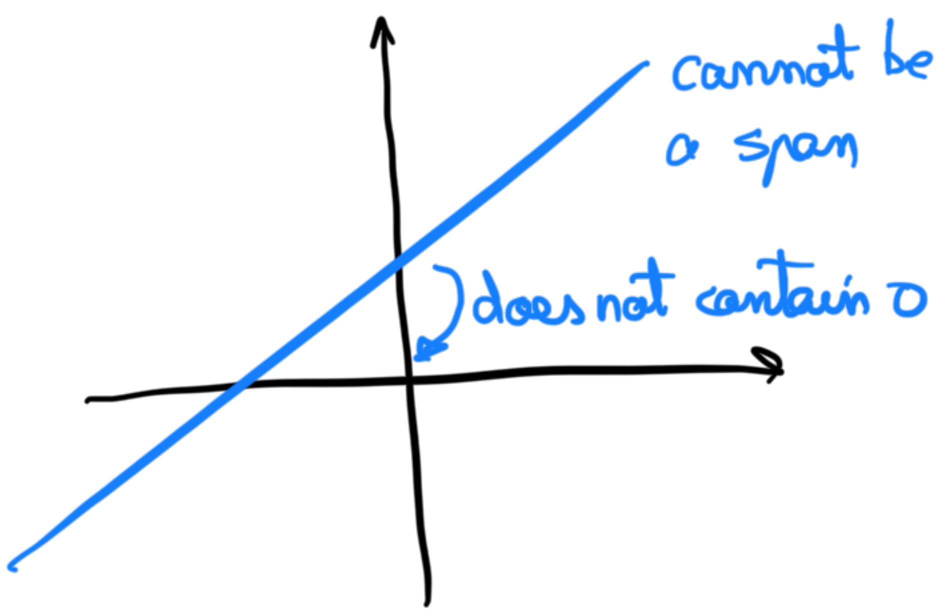


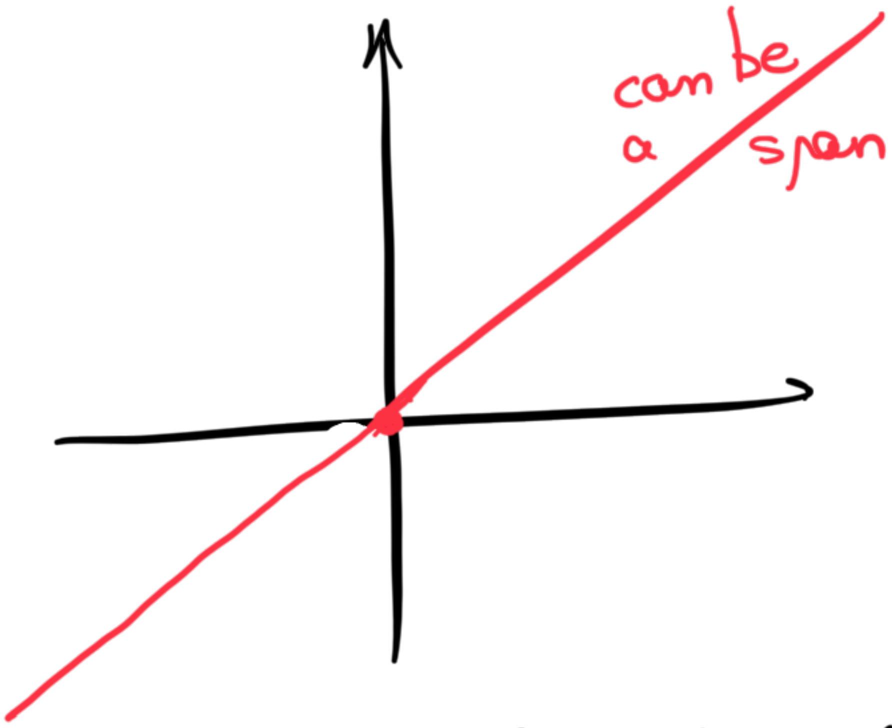
- $m=3$, $\text{span}\{v_1,v_2,v_3\}=$ a space, or → dimension 3
  a plane, or → dimension 2
  a line, or → dimension 1
  a point → dimension 0

Moral: $\text{span}\{v_1,\dots,v_m\}$ will have $\underbrace{\text{dimension}}_{\text{to be defined}} \leq m$

Fact: $\text{span}\{v_1,\dots,v_m\}$ is a $\underbrace{\text{subspace}}_{\text{i.e. a point, line, plane, space etc which contains origin}}$, i.e. contains 0

Proof: $\underbrace{0}_{\text{numbers}}v_1+0v_2+\dots+0v_m=\underbrace{0}_{\text{vector}}$





- matrices are great for encoding systems of equations $(A|b)$

equivalent

- matrices are great for understanding linear combinations & spans

m

$$A_1, \ldots, A_n \in \mathbb{R}^m, \quad x_1, \ldots, x_n \in \mathbb{R} \rightsquigarrow x_1A_1 + \ldots + x_nA_n \in \mathbb{R}^m$$

let $A = \begin{pmatrix} A_1 & A_2 & \ldots & A_n \end{pmatrix} \in \mathbb{R}^{m\times n}$ = {set of $m\times n$ real matrices}

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$; assume $A_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, A_2 = \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, A_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$

then $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$

DEF 3.1: $Ax := x_1A_1 + \ldots + x_nA_n = x_1\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \ldots + x_n\begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix}$$

$A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n = \mathbb{R}^{n\times 1}$

must be the same

Example:

$$\begin{pmatrix} 6 & -2 & 7 \\ 9 & 0 & 8 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} = -1\begin{pmatrix} 6 \\ 9 \end{pmatrix} + 5\begin{pmatrix} -2 \\ 0 \end{pmatrix} + (-2)\begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ -9 \end{pmatrix} + \begin{pmatrix} -10 \\ 0 \end{pmatrix} + \begin{pmatrix} -14 \\ -16 \end{pmatrix} = \begin{pmatrix} -30 \\ -25 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 7 \\ 9 & 0 & 8 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 6\cdot(-1) + (-2)\cdot 5 + 7\cdot(-2) \\ 9\cdot(-1) + 0\cdot 5 + 8\cdot(-2) \end{pmatrix} = \begin{pmatrix} -30 \\ -25 \end{pmatrix}$$

matrix multiplication, to be defined next week

Rules of $\underset{\in \mathbb{R}^{m\times n}}{A} \cdot \underset{\in \mathbb{R}^{n'}}{X}$ only defined if $\boxed{n = n'}$

- $A(X+Y) = AX + AY$
- $A(\lambda x) = \lambda (Ax)$
- $A0 = 0$
- $0x = 0$

zero matrix zero vector

Proofs of formulas on the left; let

$$A = (A_1 \; \dots \; A_n)$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$A(x+Y) = A\begin{pmatrix} x_1+Y_1 \\ \vdots \\ x_n+Y_n \end{pmatrix} = (x_1+Y_1)A_1 + \dots + (x_n+Y_n)A_n$$

$$\parallel$$

$$Ax + AY = x_1A_1 + \dots + x_nA_n + Y_1A_1 + \dots + Y_nA_n$$

$$A(\lambda x) = A\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} = (\lambda x_1)A_1 + \dots + (\lambda x_n)A_n$$

$$\parallel$$

$$\lambda(Ax) = \lambda A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda\left(x_1A_1 + \dots + x_nA_n\right)$$

$A \in \mathbb{R}^{n\times n}$ is called a square matrix

Particularly important is the identity matrix

$$I_n = \begin{array}{c} \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots\; 1 \end{pmatrix} \text{ n rows} \\ \text{n columns} \end{array}$$

$$\underset{\in \mathbb{R}^n}{I_n X} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1\cdot x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 \\ 0x_1 + 1x_2 + 0x_3 + 0x_4 + 0x_5 \\ 0x_1 + 0x_2 + 1\cdot x_3 + 0x_4 + 0x_5 \\ 0x_1 + 0x_2 + 0x_3 + 1x_4 + 0x_5 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = X$$

$$\Rightarrow \boxed{I_n X = X, \; \forall x \in \mathbb{R}^n}$$

Matrix-valued equation : $\underset{\mathbb{R}^{m\times n}}{\underset{\cap}{A}}\ \underset{\mathbb{R}^{n}}{\underset{\cap}{x}} = \underset{\mathbb{R}^{m}}{\underset{\cap}{b}}$

A, b are known
x must be solved for

$\Updownarrow$

system of linear equations

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$\Updownarrow$

$$\begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$\Updownarrow$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_n \end{cases}$$

SYSTEM

Going backwards from a system to a matrix equation :

$$\begin{cases} 7x_1 + 5x_2 - 3x_3 = 8 \\ \quad\quad\ \ 2x_2 - 9x_3 = -1 \end{cases} \iff \begin{pmatrix} 7x_1 + 5x_2 - 3x_3 \\ 2x_2 - 9x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

$\Updownarrow$

$$\underbrace{\begin{pmatrix} 7 & 5 & -3 \\ 0 & 2 & -9 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 8 \\ -1 \end{pmatrix}}_{b}$$

sanity check:

\# of columns of A = \# unknowns

\# of rows of A = \# equations

For which $b \in \mathbb{R}^m$ is the equation $Ax = b$ $\underbrace{\text{consistent}}_{\text{i.e. has} \geq 1 \text{ solution } x}$?

$\Uparrow$

$\Downarrow$ just language

Precisely when $b \in \operatorname{span}\{A_1, \dots, A_n\}$

notation
$$A = (A_1\ A_2 \dots A_n)$$
$$Ax = x_1 A_1 + \dots + x_n A_n$$

THM 3.2: $A \in \mathbb{R}^{m \times n}$. The equation $Ax = b$ is consistent $\forall b \in \mathbb{R}^m$

$(\iff \operatorname{span}\{A_1, \dots A_n\} = \mathbb{R}^m)$

if and only if the REF of A has no all-zero rows

if and only if some echelon form of A ——— " ———

Proof: remember that getting from $A = (A_1 \dots A_n)$ to its reduced echelon form $A^{REF} = (A_1^{REF} \dots A_n^{REF})$ is via row operations

Claim: a row operation does not change the "size" of the span of the columns of a matrix (to be proved later)

implies

$$\operatorname{span}\{A_1, \dots A_n\} = \mathbb{R}^m \iff \operatorname{span}\{A_1^{REF}, \dots A_n^{REF}\} = \mathbb{R}^m$$

- if $A^{REF} = \begin{pmatrix} 1 & * & 0 & * & * & 0 \\ 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ has an all-zero row, can $\operatorname{span}\{A_1^{REF}, \dots, A_n^{REF}\}$ equal $\mathbb{R}^m$?

No! This is because any linear combination of the columns of A^{REF} will have a 0 at the bottom, so $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^m$ cannot be in the span of the columns

- if $A^{REF} = \begin{pmatrix} 1 & * & * & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \end{pmatrix}$ has no non-zero row, can $\operatorname{span}\{A_1^{REF}, \dots, A_n^{REF}\}$ equal $\mathbb{R}^m$?

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Yes! This is because the pivot columns of A^{REF} are the "basis" vectors of $\mathbb{R}^m$, depicted in yellow

Indeed, $\forall\ x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$, we have

$$x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \dots + x_m \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{pmatrix}$$

Proof of green claim: row operations are reversible so the "size" of a set of vectors is the same before/after the operation

① swap two entries: $\begin{pmatrix} \vdots \\ b_i \\ \vdots \\ b_j \\ \vdots \end{pmatrix} \rightleftarrows \begin{pmatrix} \vdots \\ b_j \\ \vdots \\ b_i \\ \vdots \end{pmatrix}$

② rescale a row by $\lambda \neq 0$: $\begin{pmatrix} \vdots \\ b_i \\ \vdots \end{pmatrix} \rightleftarrows \begin{pmatrix} \vdots \\ \lambda b_i \\ \vdots \end{pmatrix}$

③ add $\lambda \cdot$ row i to row j $\begin{pmatrix} \vdots \\ b_i \\ \vdots \\ b_j \\ \vdots \end{pmatrix} \rightleftarrows \begin{pmatrix} \vdots \\ b_i \\ \vdots \\ b_j + \lambda b_i \\ \vdots \end{pmatrix}$

(example: the set $\mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, x, y \in \mathbb{R} \right\}$ is the same as the set $\mathbb{R}^2 = \left\{ \begin{pmatrix} 2x \\ y \end{pmatrix}, x, y \in \mathbb{R} \right\}$